On the regularity and uniqueness of the Leray weak solution to the incompressible Navier-Stokes equations in $\mathbb{R}^3$

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Abstract: In this Note we prove that the Leray turbulent solution to the three-dimensional, non-stationary, incompressible Navier-Stokes equations does not admit epochs of irregularity and is consequently smooth and unique.

Résumé: Dans cette Note on démontre que la solution turbulente de Leray pour les équations tridimensionelles instationnaires, incompressibles de Navier-Stokes n'admet pas des époques d' irrégularité et est par conséquent régulière et unique.

Version française abrégée

L'écoulement instationnaire tridimensionnel d'un fluide visqueux incompressible dans $\mathbb{R}^3$ conduit au problème de Cauchy (1)-(3). En 1934 Leray [5] prouva l'existence et la régularité partielle - Remark 3 - d'une solution faible $\underset{\sim}{v}$ de (1)-(3) ayant les propriétés (i)-(v) - voir aussi Hopf [3] - En 2001 Foias, Manley, Rosa et Temam [2] indiquèrent que la distribution $\frac{d}{dt}\left\|\underset{\sim}{v}(\cdot,t)\right\|_{L^2}^2$ est une mesure signée et bornée. En 2002 Seregin et Šverák [11] prouvèrent que si la pression p est bornée inférieurement, alors la solution faible $\underset{\sim}{v}$ est régulière et unique.

Dans la présente Note nous obtenons les résultats suivants:

Théorème 1: *La mesure signée et bornée* $\frac{d}{dt}\left\|\underset{\sim}{v}(\cdot,t)\right\|_{L^2}^2$ *est une fonction continue. Par conséquent la solution turbulente de Leray de (1)-(3) n' admet pas des époques d' irrégularité.*

Théorème 2: *La pression p donnée par (6) est bornée. Par conséquent la solution turbulente* $\underset{\sim}{v}$ *de (1)-(3) obtenue par Leray est régulière et unique.*

1. Introduction

The non-stationary motion of a viscous, incompressible fluid in $\mathbb{R}^3$ leads to the following Cauchy problem for the Navier Stokes equations:

(1) $\quad Q_T : \frac{\partial \underset{\sim}{v}}{\partial t} + \left(\underset{\sim}{v}\cdot\nabla\right)\underset{\sim}{v} - \nu\Delta\underset{\sim}{v} + \nabla p = \underset{\sim}{0}\ ,$

(2) $\quad Q_T : \nabla\cdot\underset{\sim}{v} \ = \ 0\ ,$

(3) $\quad \mathbb{R}^3 : \underset{\sim}{v}(x,0) = \underset{\sim}{v_o}(x).$

Here T is a positive number and $Q_T = \mathbb{R}^3 \times (0,T)$. By $\underset{\sim}{v}$ and p we denote the velocity and respectively the pressure. $\nu > 0$ is the viscosity coefficient and the constant density ρ is normalized to unity.
We use the following function spaces: $D = \{\underset{\sim}{\psi} \in C_o^\infty(\mathbb{R}^3)^3 \mid \nabla \cdot \underset{\sim}{\psi} = 0$ in $\mathbb{R}^3\}$, L^2_σ is the completion of D in $L^2(\mathbb{R}^3)^3$, H^1_σ is the completion of D in $W^{1,2}(\mathbb{R}^3)^3$, $L^{a,b}(Q_T) = L^a\left(0,T;\ L^b(\mathbb{R}^3)\right)$. In 1934 Leray [6] proved that for $\underset{\sim}{v_o} \in L^2_\sigma$, the Cauchy problem (1)-(3) has a weak solution - see also Hopf [3] -. This means that there exists at least one function $\underset{\sim}{v}$ having the properties:

(i) $\quad \underset{\sim}{v} \in L^2(0,T;H^1_\sigma) \cap L^\infty(0,T;L^2_\sigma)$;

(ii) $\quad$ the function $t \mapsto \int_{\mathbb{R}^3} \underset{\sim}{v}(x,t) \cdot \underset{\sim}{w}(x) d^3x$ is continuous on $[0,T)$ for all $\underset{\sim}{w} \in L^2(\mathbb{R}^3)^3$;

(iii) $$\int_0^T \int_{\mathbb{R}^3} \left[\underset{\sim}{v} \cdot \frac{\partial \underset{\sim}{\varphi}}{\partial t} - \left(\underset{\sim}{v} \cdot \nabla\right) \underset{\sim}{v} \cdot \underset{\sim}{\varphi} - \nu \nabla \underset{\sim}{v} \cdot \nabla \underset{\sim}{\varphi} \right] d^3x dt = - \int_{\mathbb{R}^3} \underset{\sim}{v_o} \cdot \underset{\sim}{\varphi}(x,0) d^3x$$

for all $\underset{\sim}{\varphi} \in C_o^\infty\left(\mathbb{R}^3 \times [0,T)\right)^3$ such that $\nabla \cdot \underset{\sim}{\varphi} = 0$ in $\mathbb{R}^3 \times [0,T]$.

(iv) $$\lim_{t \to 0} \left\| \underset{\sim}{v}(\cdot,t) - \underset{\sim}{v_o}(\cdot) \right\|_{L^2} = 0;$$

(v) $$[0,T] : \left\| \underset{\sim}{v}(\cdot,t) \right\|^2_{L^2} + 2\nu \int_0^t \left\| \nabla \underset{\sim}{v}(\cdot,s) \right\|^2_{L^2} ds \le \left\| \underset{\sim}{v_o} \right\|^2_{L^2}.$$

Remark 1 [2]: In 2001 Foias, Manley, Rosa and Temam noted that instead of (v) the following inequality holds:

(v') $$\int_0^T \int_{\mathbb{R}^3} \left[-\left| \underset{\sim}{v}(x,t) \right|^2 \psi'(t) + 2\nu \left| \nabla \underset{\sim}{v}(x,t) \right|^2 \psi(t) \right] d^3x dt \le \psi(0) \left\| \underset{\sim}{v_o} \right\|^2_{L^2}.$$

$\forall\ \psi$ real, nonnegative C^1 function on $[0,T]$, $\psi(T) = 0$.

Moreover, (v') implies that the following differential inequality in the distribution sense does hold on $(0,T)$ - see also [7] -:

(4) $$\frac{d}{dt} \left\| \underset{\sim}{v}(\cdot,t) \right\|^2_{L^2} + 2\nu \left\| \nabla \underset{\sim}{v}(\cdot,t) \right\|^2_{L^2} \le 0\ .$$

Finally they pointed out that (4) together with a deep result of Schwartz [10] implies that $\frac{d}{dt} \left\| \underset{\sim}{v}(\cdot,t) \right\|^2_{L^2}$ is a bounded signed measure, i.e. of the form

$\alpha\mu_1 - \beta\mu_2$, where μ_1, μ_2 are probability measures and $\alpha, \beta \geq 0$, and that $\left\| \underset{\sim}{v}(\cdot, t) \right\|_{L^2}^2$ is a function of bounded variation.

Remark 2 [11]: In the above formulation no information about the pressure is given. However, it can be proved -[1],[5]- that there exists a function $p \in L^{1,loc}(\mathbb{R}^3)$ for all $t \in [0, T)$, such that, for all $\delta \in (0, T)$, $\frac{3}{s} + \frac{2}{\ell} \geq 4$,

$$\bigtriangledown p, \frac{\partial \underset{\sim}{v}}{\partial t} \in L^s\left(\delta, T; L^\ell\left(\mathbb{R}^3\right)^3\right), \ \nabla^2 \underset{\sim}{v} \ \in L^s\left(\delta, T; L^\ell\left(\mathbb{R}^3\right)^{27}\right).$$

Moreover, the equation (1) holds a.e. in Q_T. The pressure p, solution of the Poisson equation

(5) $$Q_T : \triangle p = -\nabla \cdot [(\underset{\sim}{v} \cdot \nabla)\underset{\sim}{v}] \ ,$$

is determined up to an arbitrary function of t. We fix a representative for p, called normalized pressure, by setting

(6) $$Q_T : p(x,t) = \frac{1}{4\pi} \int\limits_{\mathbb{R}^3} \frac{1}{|x-y|} \nabla \cdot [(\underset{\sim}{v}(y,t) \cdot \nabla)\underset{\sim}{v}(y,t)] d^3y$$

In 2002 Seregin and Šverák [11] showed that if a weak solution to (1)-(3) develops a singularity, then the normalized pressure becomes unbounded from below. In order to do it they introduced the

Condition (C): A function $g : Q_T \to [0, \infty)$ satisfies the condition (C) if, for any $t_0 > 0$, there exists a positive number $R_0 = R_0(t_0)$ such that

(i) $$A(t_0) := \sup_{x_0 \in \mathbb{R}^3} \ \sup_{t_0 - R_0^2 \leq t \leq t_0} \ \int\limits_{B(x_0, R_0)} \frac{g(x,t)}{|x-x_0|} d^3x < \infty \ ,$$

where $B(x_0, R_0)$ is the ball with center x_0 and radius R_0, and

(ii) for each fixed $x_0 \in \mathbb{R}^3$ and each fixed $R \in (0, R_0]$, the function $t \mapsto \int\limits_{B(x_0,R)} \frac{g(x,t)}{|x-x_0|} d^3x$ is continuous at t_0 from the left.

Then, they proved the following

Theorem of Seregin and Šverák : *Let $\underset{\sim}{v}$ be a Leray-Hopf weak solution to the Cauchy problem (1)-(3), where $\underset{\sim}{v_o} \in H_\sigma^1$, and let p be the normalized pressure associated with $\underset{\sim}{v}$. Assume that there exists a function g* satisfying *the condition (C), such that*

(7) $\quad Q_T : \left| \underset{\sim}{v}(x,t) \right|^2 + 2p(x,t) \leq g(x,t)$,

or

(8) $\quad Q_T : p(x,t) \geq -g(x,t)$.

Then $\underset{\sim}{v}$ *is Hölder continuous on* Q_T *and therefore smooth and unique.*

Remark 3 [6]: Leray showed that a weak solution $\underset{\sim}{v}$ satisfies not only (v) but also, for a.a. $\sigma \geq 0$, $\forall\, t \in [\sigma, T]$ the strong energy inequality:

(v$_s$) $\quad [0,T] : \left\| \underset{\sim}{v}(\cdot,t) \right\|_{L^2}^2 + 2\nu \int_\sigma^t \left\| \nabla \underset{\sim}{v}(\cdot,s) \right\|_{L^2}^2 ds \leq \left\| \underset{\sim}{v}(\cdot,\sigma) \right\|_{L^2}^2$.

Furthermore, he proved the "partial regularity" of a weak solution $\underset{\sim}{v}$. He called thus a weak solution a turbulent one.

Theorem of Structure: *Assume* $\underset{\sim}{v}$ *is a weak solution in* Q_T *, for all* $T >$ 0*,corresponding to a volume force* $\underset{\sim}{f} \equiv 0$ *and satisfying the strong energy inequality* (v_s). *Then there exists a union* Υ *of disjoint open time intervals such that:*

(i) *The Lebesgue measure of* $(0,\infty)\backslash\Upsilon$ *is zero;*

(ii) $\underset{\sim}{v}$ *is of class* C^∞ *in* $\overline{\mathbb{R}^3} \times \Upsilon$;

(iii) *There exists* $T^* \in (0,\infty)$ *such that* $\Upsilon \supset (T^*, \infty)$;

(iv) *If* $\underset{\sim}{v_0} \in H^1$, *then* $\Upsilon \supset (0, T_1)$, *for some* $T_1 > 0$.

Leray introduced also the notion of "epoch of irregularity". A turbulent solution $\underset{\sim}{v}$ becomes irregular at the time t_1, called epoch of irregularity, iff

(a) t_1 is finite;

(b) $\underset{\sim}{v} \in C^\infty(\overline{\mathbb{R}^3} \times (t_0, t_1))$ for some $t_0 < t_1$;

(c) It is not possible to extend $\underset{\sim}{v}$ to a regular solution in (t_0, t') with $t' > t_1$.

Concerning the behaviour of $\underset{\sim}{v}$ at such a t_1 he showed that :

$\left\| \nabla \underset{\sim}{v}(\cdot,t) \right\|_{L^2}$ diverges as $t \to t_1^-$ in such a way that

(9) $\quad t < t_1 : \left\| \nabla \underset{\sim}{v}(\cdot,t) \right\|_{L^2} \geq \frac{C\nu^{3/4}}{(t_1-t)^{1/4}}$

for some positive constant C. Hence, no epoch of irregularity t_1 can occur, provided

(10) $\quad \nabla \underset{\sim}{v} \in L^4(0,T;L^2(\mathbb{R}^3))$

2. Regularity and uniqueness of the Leray weak solution.

We are now able to prove the following

Theorem 1: *The bounded signed measure* $\frac{d}{dt}\left\|\underset{\sim}{v}(\cdot,t)\right\|_{L^2}^2$ *is a continuous function. Therefore the Leray turbulent solution* $\underset{\sim}{v}$ *of (1)-(3) does not admit epochs of irregularity.*

For the proof we need the following results:

Definition of quasi-uniform convergence [9, p. 66]: *Let X and Y be metric spaces and let f_n, $n \in \mathbb{N}$, map X into Y. The sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to converge quasi-uniformly on X to $f : X \to Y$ if (i) $\{f_n\}$ converges pointwise to f; (ii) for every $\varepsilon > 0$ there exists a finite or infinite, increasing sequence $\{n_p\}_{p\in\mathbb{N}} \subset \mathbb{N}$ and the sequence $\{D_p\}_{p\in\mathbb{N}}$ of open sets $D_p \subset X$, $X = \bigcup_{p=1}^{\infty} D_p$, such that $dist_Y(f(x), f_{n_p}(x)) < \varepsilon$, $p \in \mathbb{N}$, $x \in D_p$.*

Theorem of Arzelà, Hobson, Borel, Gagaeff and Alexandrov [9, p. 68]. *Let X and Y be metric spaces and let f_n, $n \in \mathbb{N}$, map X into Y continuously. The sequence $\{f_n\}$ converges on X to a continuous map $f : X \to Y$, iff the convergence is quasi-uniform.*

Proof of the Lemma: To this end let us consider the Cauchy problem for the regularized Navier-Stokes equations [1],[7],[8]:

(11) $\quad Q_T : \frac{\partial \underset{\sim}{v}^\varepsilon}{\partial t} + J_\varepsilon[(J_\varepsilon \underset{\sim}{v}^\varepsilon)\cdot\nabla(J_\varepsilon \underset{\sim}{v}^\varepsilon)] + \nabla p^\varepsilon - \nu J_\varepsilon(J_\varepsilon \Delta \underset{\sim}{v}^\varepsilon) = \underset{\sim}{0},$

(12) $\quad Q_T : \nabla \cdot \underset{\sim}{v}^\varepsilon = 0 \ ,$

(13) $\quad \mathbb{R}^3 : \ \underset{\sim}{v}^\varepsilon(x,0) = \underset{\sim 0}{v}(x) \ ,$

where $J_\varepsilon \underset{\sim}{v}$ is the mollification of $\underset{\sim}{v}$, i.e.

(14) $\quad \mathbb{R}^3 : (J_\varepsilon \underset{\sim}{v})(x) = \varepsilon^{-3} \int_{\mathbb{R}^3} \rho(\frac{x-y}{\varepsilon}) \underset{\sim}{v}(y) d^3y \ , \quad \varepsilon > 0 \ ,$

with the radial function $\rho(|x|) \in C_o^\infty(\mathbb{R}^3)$, $\rho \geq 0$, $\int_{\mathbb{R}^3} \rho d^3x = 1$.

As known, [8, p. 102] (11)-(13) has a unique solution $\underset{\sim}{v}^{\varepsilon} \in C^1([0,T);H^1_\sigma)$, provided $\underset{\sim 0}{v} \in H^1_\sigma$. Moreover the following energy equality does hold:

$$(15) \qquad (0,T) : \tfrac{d}{dt}\left\|\underset{\sim}{v}^{\varepsilon}(\cdot,t)\right\|^2_{L^2} + 2\nu\left\|\bigtriangledown J_\varepsilon \underset{\sim}{v}^{\varepsilon}(\cdot,t)\right\|^2_{L^2} = 0 \ .$$

From (15) we get then

$$(16) \qquad \sup_{t\in[0,T]}\left\|\underset{\sim}{v}^{\varepsilon}(\cdot,t)\right\|_{L^2} \le \left\|\underset{\sim 0}{v}\right\|_{L^2} .$$

Hence $\underset{\sim}{v}^{\varepsilon}$ is uniformly bounded in $L^2(0,T;L^2_\sigma)$, so there exists, according to the Banach-Alaoglu theorem, a subsequence which we denote again by $\{\underset{\sim}{v}^{\varepsilon}\}$ that converges weakly to the Leray weak solution $\underset{\sim}{v} \in L^2(0,T;L^2_\sigma)$. Moreover this convergence is even a strong one. -[12]- As noted in Remark 1, $\left\|\underset{\sim}{v}(\cdot,t)\right\|^2_{L^2}$ is a function of bounded variation and, in accordance with [4, p. 343], has the unique (up to a constant) decomposition:

$$(17) \qquad [0,\infty) : \left\|\underset{\sim}{v}(\cdot,t)\right\|^2_{L^2} = \varphi(t) + H(t) + \psi(t).$$

Here φ is its absolutely continuous part, H is the function of its jumps and ψ is its singular part, i.e. ψ is a continuous function of bounded variation. Moreover, the classical derivatives of H and ψ vanish almost everywhere. We note next the very important fact that the distribution derivative and the classical derivative of a function f of bounded variation are equal to each other only in the case where f is absolutely continuous [4, p. 343]. Let us now show that $\left\|\underset{\sim}{v}(\cdot,t)\right\|^2_{L^2}$ does not have jumps. To this end we assume by contradiction that $\left\|\underset{\sim}{v}(\cdot,t)\right\|^2_{L^2}$ has at $t = t_0$ a jump h. By integrating (15) with respect to t and applying the mean value theorem in integral form we obtain

$$(18) \qquad \left\|\underset{\sim}{v}^{\varepsilon}(\cdot,t)\right\|^2_{L^2} - \left\|\underset{\sim}{v}^{\varepsilon}(\cdot,t_0)\right\|^2_{L^2} = -2\nu(t-t_0)\left\|\bigtriangledown J_\varepsilon \underset{\sim}{v}^{\varepsilon}(\cdot,t_0+\delta(t-t_0))\right\|^2_{L^2},$$

where $0 \le \delta \le 1$ may depend on ε. Letting now $\varepsilon \to 0$ we get on one hand

$$(19) \qquad [0,T] : \lim_{\varepsilon\to 0}\left\|\bigtriangledown J_\varepsilon \underset{\sim}{v}(\cdot,t_0+\delta(t-t_0))\right\|^2_{L^2} = -\frac{\left\|\underset{\sim}{v}(\cdot,t)\right\|^2_{L^2} - \left\|\underset{\sim}{v}(\cdot,t_0)\right\|^2_{L^2}}{2\nu(t-t_0)}.$$

On the other hand $\left\|\underset{\sim}{v}(\cdot,t)\right\|_{L^2}^2$ has at $t=t_0$ the jump h. From (19) it follows then that for ε near 0 the function $\left\|\nabla J_\varepsilon \underset{\sim}{v}^\varepsilon(\cdot,t)\right\|_{L^2}^2$ has near $t=t_0$ a singularity, in contradiction with its continuity. Hence, according to (17), $\left\|\underset{\sim}{v}(\cdot,t)\right\|_{L^2}^2$ is a continuous function of bounded variation. We get then on one hand for all $g \in L^2(0,\infty)$

$$(20) \qquad [0,\infty) : \lim_{\varepsilon\to 0} \int_0^t \left\|\underset{\sim}{v}^\varepsilon(\cdot,s)\right\|_{L^2}^2 g(s)ds = \int_0^t \left\|\underset{\sim}{v}(\cdot,s)\right\|_{L^2}^2 g(s)ds.$$

On the other hand for all $h \in C[0,\infty)$ it follows

$$(21) \qquad [0,\infty) : \lim_{\delta\to 0}(J_\delta h)(t) = h(t),$$

where $J_\delta h$ is the mollification of h. Both convergences (20) and (21) are, according to the Theorem of Arzelà, Hobson, Borel, Gagaeff and Alexandrov, quasi-uniform ones. From (20) and (21) we infer that the sequence $\left\{\left\|\underset{\sim}{v}^\varepsilon(\cdot,t)\right\|_{L^2}^2\right\}$ converges quasi-uniformly on $(0,\infty)$ to $\left\|\underset{\sim}{v}(\cdot,t)\right\|_{L^2}^2$. Moreover, taking into account for $t_0,\ t \in (0,\infty)$ the quasi-uniform convergence

$$(22) \qquad \lim_{\varepsilon\to 0}(t-t_0)^{-1} \int_{t_0}^t \frac{d}{ds} \left\|\underset{\sim}{v}^\varepsilon(\cdot,s)\right\|_{L^2}^2 ds = \frac{\left\|\underset{\sim}{v}(\cdot,t)\right\|_{L^2}^2 - \left\|\underset{\sim}{v}(\cdot,t_0)\right\|_{L^2}^2}{t-t_0}$$

as well as the fact that the right hand-side of (22) has for t tending to t_0 a finite limit almost everywhere on $(0,\infty)$, we get that also the sequence $\left\{\frac{d}{dt}\left\|\underset{\sim}{v}^\varepsilon(\cdot,t)\right\|_{L^2}^2\right\}$ converges quasi-uniformly to $\frac{d}{dt}\left\|\underset{\sim}{v}(\cdot,t)\right\|_{L^2}^2$ and the assertion of the Lemma follows.

We can now prove the following

Theorem 2: *The normalized pressure p given by (6) is bounded by a constant function $g(x,t)$. Since a constant function $g(x,t)$ satisfies the condition (C), the weak Leray solution $\underset{\sim}{v}$ of (1)-(3), where $\underset{\sim}{v}_o \in H^1_\sigma$, is smooth and unique.*

Proof of Theorem 2.- Let us first note that

$$(23) \qquad \nabla \cdot [(\underset{\sim}{v}\cdot\nabla)\underset{\sim}{v}] \le \left|\nabla \underset{\sim}{v}\right|^2 \qquad \text{a.e.in}\, Q_T\ .$$

Using (23), from (6) it follows then

$$(24)\qquad Q_T : |p(x,t)| \le \frac{1}{4\pi} \int\limits_{\mathbb{R}^3} |x-y|^{-1} \left| \nabla \cdot [(\underset{\sim}{v}(y,t)\cdot\nabla)\underset{\sim}{v}(y,t)] \right| d^3y \le$$

$$\frac{1}{4\pi}\left[\int\limits_{|x-y|\le 1} |x-y|^{-1} \left|\nabla \underset{\sim}{v}(y,t)\right|^2 d^3y + \int\limits_{|x-y|>1} |x-y|^{-1} \left|\nabla \underset{\sim}{v}(y,t)\right|^2 d^3y \right] \le$$

$$\frac{1}{4\pi}\left[\left\|\nabla \underset{\sim}{v}(\cdot,t)\right\|_{L^2}^2 + \int\limits_{|z|\le 1} |z|^{-1} \left|\nabla \underset{\sim}{v}(x+z,t)\right|^2 d^3z \right].$$

It remains only to estimate the last integral, i.e.

$$(25)\qquad Q_T : \int\limits_{|z|\le 1} |z|^{-1} \left|\nabla \underset{\sim}{v}(x+z,t)\right|^2 d^3z = \int\limits_{r\le 1} r \left|\nabla \underset{\sim}{v}\right|^2 \sin\varphi dr d\varphi d\theta,$$

where r, φ and θ are the spherical coordinates. To this end let $\varepsilon \in (0,1)$ be an arbitrarily chosen but fixed number. Since according to (15) and the Theorem 1, $\left\|\nabla \underset{\sim}{v}(\cdot,t)\right\|_{L^2}$ is a bounded function, from the estimate

$$(26)\qquad \begin{aligned} Q_T : \varepsilon \int_{\varepsilon\le r\le 1} r \left|\nabla \underset{\sim}{v}\right|^2 \sin\varphi \; dr \; d\varphi \; d\theta \;&\le\; \int_{\varepsilon\le r\le 1} r^2 \left|\nabla \underset{\sim}{v}\right|^2 \sin\varphi \; dr \; d\varphi \; d\theta \\ &\le\; \left\|\nabla \underset{\sim}{v}(\cdot,t)\right\|_{L^2}^2 \end{aligned}$$

it follows immediately that the only possible non-integrable singularity of $|z|^{-1}\left|\nabla \underset{\sim}{v}(x+z,t)\right|^2$ in the ball $|z|\le 1$ is its center $z=0$. In order to prove the contrary, we use the results of Leray concerning the "partial regularity" of a weak solution $\underset{\sim}{v}$. According to these results, $|z|^{-1}\left|\nabla \underset{\sim}{v}(x+z,t)\right|^2$ has no non-integrable singularity at the center $z=0$ of the ball $|z|\le 1$, provided t is not an epoch of irregularity. On the other hand, since $\left\|\nabla \underset{\sim}{v}(\cdot,t)\right\|_{L^2}$ is bounded, no epoch of irregularity can occur. Taking now into account the fact that $\underset{\sim}{v}$ and its derivatives vanish for $|x|\to\infty$, provided t is not an epoch of irregularity, from (24), (25) and (26) we get then

$$(27)\qquad Q_T : |p(x,t)| \le \frac{\tilde{C}+1}{4\pi} \left\|\nabla \underset{\sim}{v}(\cdot,t)\right\|_{L^2}^2 ,$$

where the positive constant $\tilde{C}$ can be chosen to be independent of x and t. Indeed, denote to this end by t_1 the first possible epoch of irregularity, and let $t_0 > 0$ be arbitrarily chosen. Since $\underset{\sim}{v} \in C^\infty(\overline{\mathbb{R}^3 \times [t_0, t_1-\delta]})$ for a fixed but arbitrarily chosen $\delta > 0$, we have on one hand:

(28) $\quad \lim_{|x|\to\infty} \max_{[t_o,t_1-\delta]} \int_{|z|\leq\varepsilon} r \left|\nabla \underset{\sim}{v}(x+z,t)\right|^2 \sin\varphi dr d\varphi d\theta = 0.$

On the other hand by the integral theorem of the mean

(29)
$$K \times [t_0, t_1-\delta] \ : \ \int_{|z|\leq\varepsilon} r \left|\nabla \underset{\sim}{v}(x+z,t)\right|^2 \sin\varphi dr d\varphi d\theta =$$
$$\pi\varepsilon^2 \left|\nabla \underset{\sim}{v}(x+z^*,t)\right|^2 \leq \pi\varepsilon^2 \max_{K\times[t_o,t_1-\delta]} \left|\nabla \underset{\sim}{v}\right|^2 \ ,$$

(30)
$$K \times [t_0, t_1-\delta] \ : \ \int_{|z|\leq\varepsilon} r^2 \left|\nabla \underset{\sim}{v}(x+z,t)\right|^2 \sin\varphi dr d\varphi d\theta =$$
$$C^* \int_{|z|\leq\varepsilon} r \left|\nabla \underset{\sim}{v}(x+z,t)\right|^2 \sin\varphi dr d\varphi d\theta$$

where $|z^*| \leq \varepsilon$, K is a compact set and $0 < C^* \leq \varepsilon$. From (28), (29) and (30) we infer then

(31) $\quad \mathbb{R}^3 \times [t_0, t_1-\delta] \ : \ \int_{|z|\leq\varepsilon} r \left|\nabla \underset{\sim}{v}\right|^2 \sin\varphi dr d\varphi d\theta \leq C_1 \left\|\nabla \underset{\sim}{v}(\cdot,t)\right\|_{L^2}^2 \ ,$

with C_1 independent of x and t.

Letting now $\delta \to 0$ in (29), using again the fact that $\left\|\nabla \underset{\sim}{v}(\cdot,t)\right\|_{L^2}$ is bounded, and repeating the same argument for all possible epochs of irregularity, we conclude the validity of (27) with $\tilde{C}$ independent of x and t, and, consequently, the boundedness of $p(x,t)$ in Q_T. The assumptions of the Theorem of Seregin and Šverák being satisfied, the Leray turbulent solution $\underset{\sim}{v}$ is smooth and unique.

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